MATHEMATICS SPECIALIST

MAWA Year 12 Examination 2018

Calculator-free

Marking Key

The release date for this exam and marking scheme is

- **the end of week 1 of term 4, 2018**

Question 1 **(5 marks)**

Solution
$\lvert z\rvert^4 = (-1)^2 + (\sqrt{3})^2 = 4.$ Hence $\lvert z\rvert = \sqrt{2}$ Also the argument of z lies in the fourth quadrant with $\arg z^4 = \tan^{-1}\left(-\sqrt{3}\right) = 2\pi - \tan^{-1}\left(\sqrt{3}\right) = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ Thus $z^4 = 4\exp\left(\frac{5\pi}{3} + 2k\pi\right)$ for integer $k = 0,1,2,3$ so $z = \sqrt{2}\exp\left(\frac{5\pi}{12} + \frac{k\pi}{2}\right)$ Hence the four solutions are $z = \sqrt{2}\exp(i\vartheta)$ where $\vartheta = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}$ and $\frac{23\pi}{12}$. Restricting to the given range requires that $z = \sqrt{2}\exp(i\vartheta)$ where $\vartheta = -\frac{7\pi}{12}, -\frac{\pi}{12}, \frac{5\pi}{12}$ and $\frac{11\pi}{12}$.

Mathematical behaviours	Marks
• states the correct value for $\lvert z\rvert$	1
• gives the correct value for $\arg z^4$	1
• calculates four distinct solutions of the equation (one mark for 2 or 3)	2
• restricts the arguments to the appropriate range	1

Question 2 (a) **(4 marks)**

Solution
Since $\cos 2x = 2\cos^2 x - 1$ it follows that $$\int_{\pi/6}^{\pi/4} \frac{dx}{1+\cos 2x} = \frac{1}{2}\int_{\pi/6}^{\pi/4} \sec^2 x \ dx = \frac{1}{2}\left[\tan x\right]_{\pi/6}^{\pi/4} = \frac{1}{2}\left(1-\frac{1}{\sqrt{3}}\right).$$

Mathematical behaviours	Marks
• identifies correct double angle formula to use	1
• simplifies the integral to requiring the anti-derivative of $\sec^2 x$	1
• integrates correctly	1
• evaluates the indefinite integral at the end points	1

Question 2(b) **(4 marks)**

Solution
If we put $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2u}$ then we find that $$\int_4^9 \frac{dx}{x+\sqrt{x}} = 2\int_2^3 \frac{u}{u^2+u}\ du = 2\int_2^3 \frac{du}{1+u} = 2\left[\ln(1+u)\right]_{u=2}^{u=3}$$ $$= 2[\ln 4 - \ln 3] = 2\ln(\tfrac{4}{3})$$

Mathematical behaviours	Marks
• calculates du/dx correctly	1
• substitutes into integral changing the limits appropriately	1
• integrates the expression correctly	1
• substitutes the boundary values and simplifies to a suitable form	1

Question 2 (c) **(4 marks)**

Solution	
If we put $v=\cos x$ then $\frac{dv}{dx}=-\sin x$ and the integral $$\int_0^{\pi/2}\sin x\cos^n x\,dx=-\int_1^0 v^n\,dv=\int_0^1 v^n dv=\frac{1}{n+1}$$ Hence if the integral equals $\frac{1}{2018}$ we conclude that $n=2017$	
Mathematical behaviours	**Marks**
• identifies that $\int f\,'(x)[f(x)]^n dx=\frac{[f(x)]^{n+1}}{n+1}$ • identifies the most appropriate substitution • evaluates the integral correctly and thereby • deduces the correct value of n	1 1 1 1

Question 3 (a) **(2 marks)**

Solution	
Graph (A) Equation I $\frac{dy}{dx}=e^{-x}$ Graph (B) Equation III $\frac{dy}{dx}=\cos x$ Graph (C) Equation II $\frac{dy}{dx}=y-x$	
Mathematical behaviours	**Marks**
• matches one graph correctly • matches a second graph correctly	1 1

Question 3 (b)(i) **(3 marks)**

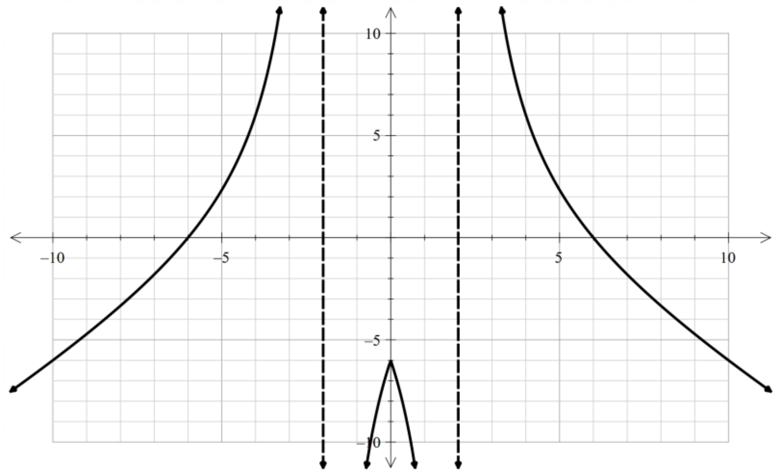
Solution	
$\frac{d^2y}{dx^2}=(y-2)^2+2x(y-2)\frac{dy}{dx}$ $x=0,\ y=-2,\ \frac{dy}{dx}=0,\ \frac{d^2y}{dx^2}>0$ Hence at $x=0$, f has a relative minimum.	
Mathematical behaviours	**Marks**
• uses implicit differentiation to determine $\frac{d^2y}{dx^2}$ • calculates $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x=0$ • correct conclusion	1 1 1

Question 3 (b)(ii) **(4 marks)**

Solution	
$\int \frac{dy}{(y-2)^2} = \int x\,dx \quad \Rightarrow \quad -\frac{1}{y-2} = \frac{x^2}{2} + c_1$ $\Rightarrow \quad y = 2 - \frac{2}{x^2 + c_2}$ $x = 0,\ y = -2 \quad \Rightarrow c_2 = \frac{1}{2}$ $f(x) = 2 - \frac{4}{2x^2 + 1}$	
Mathematical behaviours	**Marks**
• separates the variables • determines the correct anti-derivaties • calculates the constant correctly • states the required particular solution	1 1 1 1

Question 4 (a) **(3 marks)**

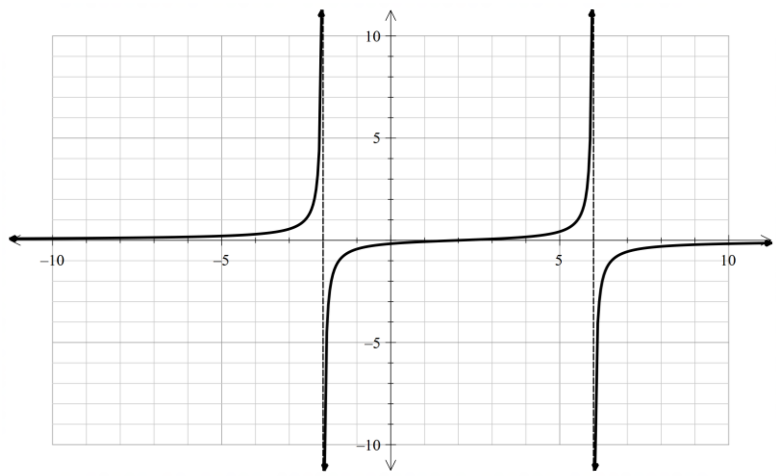
Solution
$g(x) = f(\lvert x \rvert)$



Mathematical behaviours	Marks
• draws both vertical asymptotes	1
• draws correct graph for $\lvert x \rvert > 2$	1
• draws correct graph for $-2 < x < 2$	1

Question 4 (b) **(3 marks)**

Solution
$h(x) = \dfrac{1}{f(x)}$



Mathematical behaviours	Marks
• draws vertical asymptotes at $x = -2$ and at $x = 6$	1
• draws correct graph for $-2 < x < 6$	1
• draws correct graph for $x < -2$ and for $x > 6$	1

Question 5 (a) **(2 marks)**

Solution	
The equation $x^2 + y^2 + z^2 - 8x + 12y - 24z + 171 = 0$ can be rewritten as $(x-4)^2 + (y+6)^2 + (z-12)^2 = 4^2 + 6^2 + 12^2 - 171 = 25$ (*) So the centre C has coordinates $(4, -6, 12)$ and the radius is $\sqrt{25} = 5$	
Mathematical behaviours	**Marks**
• obtains co-ordinates of C • calculates radius correctly	1 1

Question 5 (b) **(3 marks)**

Solution	
The point A lies on the line segment $\overrightarrow{\boldsymbol{OC}}$ and on the sphere S. So A has coordinates $(4t, -6t, 12t)$ for some t Substituting into the equation for S gives $(4t-4)^2 + (-6t+6)^2 + (12t-12)^2 = 25$ i.e. $196(t-1)^2 = 25$. i.e. $t - 1 = \pm 5/14$ $t = 9/14$ gives the point closest to O, so the coordinates of A are $\left(\frac{18}{7}, -\frac{27}{7}, \frac{54}{7}\right)$. Alternative method: Distance of the centre of the sphere from the origin is $\sqrt{4^2 + (-6)^2 + 12^2} = \sqrt{196} = 14$ Radius of sphere is 5 so required point is $\frac{9}{14}$ along the line joining O to $(4, -6, 12)$ Hence the point A is as before	
Mathematical behaviours	**Marks**
• obtains the correct form of the co-ordinates A in terms of a parameter • solves for the parameter • derives the appropriate co-ordinates of A ALTERNATIVE • determines distance of centre from origin • determines required point is 9/14ths along the line OC • derives the appropriate co-ordinates of A	1 1 1 1 1 1

Question 5 (c) **(3 marks)**

Solution	
The vector $\overrightarrow{\boldsymbol{OA}} = 2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}$ is normal to P. So $2x - 3y + 6z = c$ (*) is a Cartesian equation of P. Since A $\left(\frac{18}{7}, -\frac{27}{7}, \frac{54}{7}\right)$ lies on P, $c = \frac{36}{7} + \frac{81}{7} + \frac{324}{7} = \frac{441}{7} = 63$ So $2x - 3y + 6z = 63$ is a Cartesian equation of P.	
Mathematical behaviours	**Marks**
• recognises the normal to the plane • writes down the correct form of the equation of the plane (*) • evaluates the constant correctly	1 1 1

Question 6 (a) **(2 marks)**

Solution	
False The confidence interval may contain NONE of the original population. For example, if the population consists just of 0's and 1's, and the sample size is large enough, then $0 < \bar{X} - E < \bar{X} + E < 1.$	
Mathematical behaviours	**Marks**
• states correct answer • gives a valid reason	1 1

Question 6 (b) **(2 marks)**

Solution	
True The probability that any one confidence interval will contain the mean is equal to the confidence level, i.e. 90% or 0.9	
Mathematical behaviours	**Marks**
• states correct answer • gives a valid reason	1 1

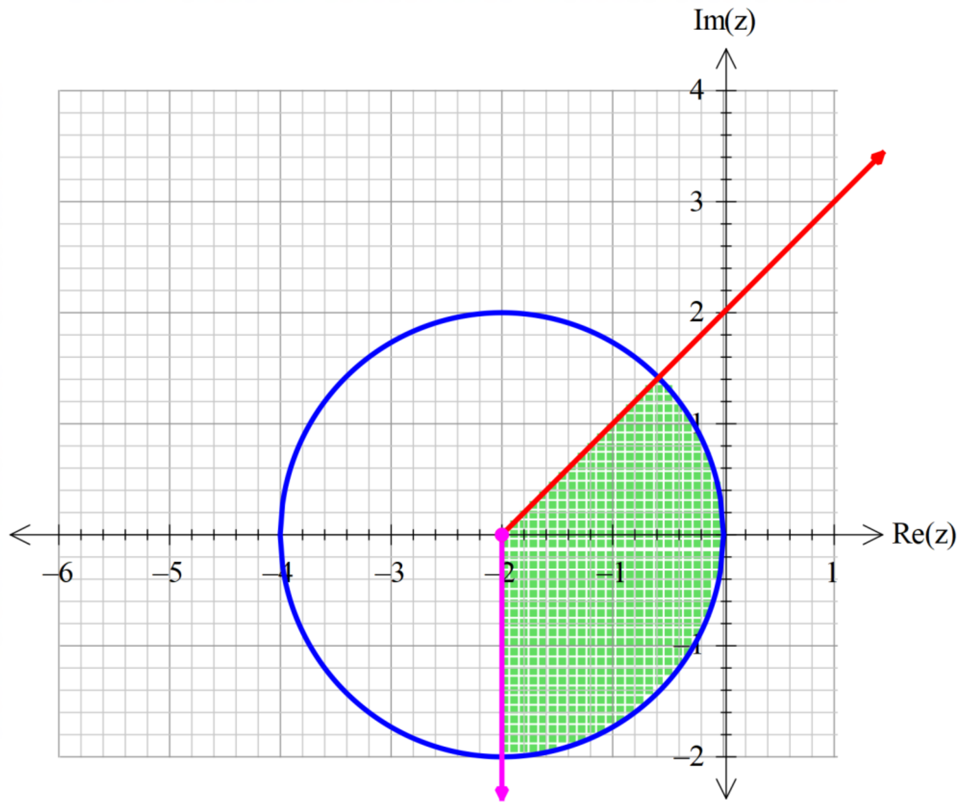
Question 6 (c) **(2 marks)**

Solution	
False Because the samples are independent and random it is possible that NONE of the confidence intervals will contain μ	
Mathematical behaviours	Marks
• states correct answer • gives a valid reason	1 1

Question 6 (d) **(3 marks)**

Solution	
True The probability that exactly 9 of the 10 confidence intervals will contain μ is $B(10,9,0.9) = \binom{10}{9} \times 0.9^9 \times 0.1^1 = 10 \times 0.9^9 \times 0.1 = 0.9^9$ (*) On the other hand, the probability that all of the 10 confidence intervals will contain μ is $B(10,10,0.9) = 0.9^{10}$. (**) Clearly $0.9^9 > 0.9^{10}$.	
Mathematical behaviours	Marks
• states correct answer • derives the correct expressions (*) and (**) for the respective probabilities	1 1+1

Question 7 **(5 marks)**

Solution	
	
Mathematical behaviours	**Marks**
• shows circle with correct centre and radius	1+1
• shows correct wedge with the correct angles	1
• shades required area indicating that boundaries should be included	1+1